

Paths with singularities in functional integrals of quantum field theory.

V.V. Belokurov and E.T. Shavgulidze
Lomonosov Moscow State University, Russia
e-mail: belokur@rector.msu.ru

The quantum field theory model studied in our recent paper [1] demonstrates that quantum properties of a model depend in a great extent on the functional space we integrate over.

Here we consider a toy model: φ^4 -interacting quantum field theory in one-dimensional "Euclidean" space-time. We prove that the functional integrals of the free field theory evaluated over the space of continuous functions are equal to the functional integrals of the interacting field theory evaluated over a set of spaces containing the spaces of discontinuous functions.

Our toy model is given by the functional integral

$$\int \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt - \frac{1}{2} \int_0^1 \varphi^4(t) dt \right\} d\varphi. \quad (1)$$

Here we suppose for simplicity that the field function $\varphi(t)$ (or stochastic process) is defined on the finite closed interval $[0, 1]$.

The integral (1) exists for continuous field functions ($\varphi(t) \in C[0, 1]$). (It is the integral over Wiener measure, the integrand $\exp\{-\frac{1}{2} \int_0^1 \varphi^4(t) dt\}$ being a bounded functional. Notice that here all the derivatives are understood in a generalized sense.)

After the substitution

$$\psi(t) = \varphi(t) + \int_0^t \varphi^2(\tau) d\tau, \quad (2)$$

formally we get

$$\dot{\psi}(t) = \dot{\varphi}(t) + \varphi^2(t),$$

and

$$\int_0^1 (\dot{\psi}(t))^2 dt = \int_0^1 (\dot{\varphi}(t))^2 dt + \int_0^1 \varphi^4(t) dt + 2 \int_0^1 (\dot{\varphi}(t)) \varphi^2(t) dt. \quad (3)$$

Let $\psi(0) = \varphi(0) = 0$. Then the last term in (3) equals to $\frac{2}{3} \varphi^3(1)$.

The discrete versions of the substitution (2) looks like

$$\psi(t_k) = \varphi(t_k) + \frac{1}{N} \sum_{i=0}^{k-1} \varphi^2(t_i)$$

and the Jakobian of this substitution $J = 1$. If we consider the functional integrals as the Ito stochastic integrals [2] we can make a conclusion about the formal equality of the integrals

$$\int \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt - \frac{1}{2} \int_0^1 \varphi^4(t) dt - \frac{1}{3} \varphi^3(1) \right\} d\varphi$$

and

$$\int \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\psi}(t))^2 dt \right\} d\psi.$$

However, so far we have not specified the functional spaces we integrate over. As we show later, for the equality of these integrals to be valid the functional spaces should be different. Namely,

$$\begin{aligned} \int_{C[0,1]} \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt - \frac{1}{2} \int_0^1 \varphi^4(t) dt - \frac{1}{3} \varphi^3(1) \right\} d\varphi = \\ \int_Y \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\psi}(t))^2 dt \right\} d\psi \neq 1. \end{aligned} \quad (4)$$

Where,

$$Y \subset C[0,1], \quad Y \neq C[0,1].$$

And vice versa,

$$\begin{aligned} 1 = \int_{C[0,1]} \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\psi}(t))^2 dt \right\} d\psi = \\ \int_X \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt - \frac{1}{2} \int_0^1 \varphi^4(t) dt - \frac{1}{3} \varphi^3(1) \right\} d\varphi. \end{aligned} \quad (5)$$

Where,

$$X \neq C[0,1], \quad X \supset C[0,1].$$

Now, let us study the structure of the space X .

Suppose we know the function $\psi(t) \in C[0,1]$. Let us find the general form of the function $\varphi(t)$ satisfying the equation (2). Consider the function $\mu(t) = \psi(t) - \varphi(t)$. We have

$$\mu(t) = \int_0^t (\mu(\tau) - \psi(\tau))^2 d\tau,$$

and

$$\dot{\mu}(t) = (\mu(t) - \psi(t))^2.$$

So, there is a point t_1 where the function $\mu(t_1) > 0$.

For the function $\nu(t) = \frac{1}{\mu(t)}$ the differential equation looks like

$$\dot{\nu}(t) = -(1 - \psi(t)\nu(t))^2.$$

As $\dot{\nu}(t) < 0$ and $\nu(t_1) > 0$ there can be a point $t^* \in [t_1, 1]$ where $\nu(t^*) = 0$.

In the vicinity of this point we have

$$\nu(t) = - \int_{t^*}^t (1 - \psi(\tau)\nu(\tau))^2 d\tau = -(t - t^*) - \psi(\tilde{t})(t - t^*)^2 + O((t - t^*)^3), \quad \tilde{t} \in [t^*, t],$$

and

$$\mu(t) = -\frac{1}{t - t^*} + \psi(\tilde{t}) + O((t - t^*)), \quad \tilde{t} \in [t^*, t].$$

That is, the function $\varphi(t)$ has a singular point and in the vicinity of the point $\varphi(t)$ is of the form

$$\varphi(t) = \frac{1}{t - t^*} + \chi(t), \quad (6)$$

where $\chi(t) \in C[0, 1]$, $\chi(t^*) = 0$.

As the function $\chi(t)$ is bounded on $[0, 1]$, the interval $[t_1, t_2] \subset [0, 1]$ where

$$\varphi(t) \simeq \frac{1}{t - t^*}$$

is finite.

Depending on the form of the function $\psi(t)$ there can be other finite intervals $[t_3, t_4], \dots$ where $\varphi(t)$ has singularities of the same type. However, due to the compactness of $[0, 1]$ the number of the singularities is finite for every $\psi(t)$.

Thus, the general form of the functions $\varphi(t)$ is

$$\varphi(t) = \sum_{i=0}^n \frac{1}{t - t_i^*} + \xi(t), \quad \xi(t) \in C[0, 1]. \quad (7)$$

The singularities from φ and $\int_0^t \varphi^2 d\tau$ cancel each other in (2) demonstrating something like renormalization of the field function [3].

Now, the space X can be represented in the form

$$X = X_0 \cup X_1 \cup X_2 \cup \dots X_n \cup \dots,$$

where $X_0 = C[0, 1]$ and X_n is the space of functions of the type (7) with n singularities.

The equation (5) looks like

$$1 = \int_{C[0,1]} \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\psi}(t))^2 dt \right\} d\psi =$$

$$\sum_{n=0}^{\infty} \int_{X_n} \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt - \frac{1}{2} \int_0^1 \varphi^4(t) dt - \frac{1}{3} \varphi^3(1) \right\} d\varphi. \quad (8)$$

We can easily evaluate [4] the generating functional

$$\begin{aligned} \int_{C[0,1]} \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\psi}(t))^2 dt + i \int_0^1 \psi(t) \eta(t) dt \right\} d\psi = \\ \exp \left\{ -\frac{1}{2} \int_0^1 \int_0^1 \min(t_1, t_2) \eta(t_1) \eta(t_2) dt_1 dt_2 \right\}. \end{aligned} \quad (9)$$

However, it is useless for evaluating the Green functions of the field φ . The integrals of the form

$$\int_{X_n} \exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt \right\} P(\varphi) d\varphi, \quad n \geq 1,$$

where $P(\varphi)$ is a polynomial, do not exist.

$$\exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt \right\} d\varphi$$

is not a measure on X_n , ($n \geq 1$). Here the sum of two terms compensating the singularities of each other is needed. And only

$$\exp \left\{ -\frac{1}{2} \int_0^1 (\dot{\varphi}(t))^2 dt - \frac{1}{2} \int_0^1 \varphi^4(t) dt \right\} d\varphi$$

can be considered as a measure on X_n , ($n \geq 1$).

Thus, in the toy model we confirm the conjecture stated by N.N. Bogoliubov many years ago [5] that in quantum field theory the expansion in powers of interaction is the reason not only of perturbation theory series divergence, but the reason of the divergence of individual terms as well.

References

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